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Impulsive Differential Systems and the Pulse Phenomena

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As is known [1, 2], the solutions of differential systems with impulses may experience the pulse phenomena, namely the solutions may hit a given surface finite or infinite number of times causing rhythmical beating. This situation presents difficulties in the investigation of properties of solutions of such systems. Consequently, it is desirable to find conditions that guarantee the absence or presence of pulse phenomena. In this paper, we shall discuss this problem.

Consider the impulsive differential system

$$\begin{aligned} x' &= f(t, x),\ t \neq \tau_k(x), \qquad & x(t_0) &= x_0,\ t_0 \geq 0 \\ \Delta x &= I_k(x), & t &= \tau_k(x), \end{aligned} \tag{1}$$

where $f \in C[R_+ \times \Omega, R^n]$, $\Omega \subset R^n$ being an open set. Let us begin with the following result which gives a simple set of sufficient conditions for the absence of pulse phenomena and shows the interplay between the functions f, τ_k, and I_k.

THEOREM 1. *Assume that*

(i) $f \in C[R_+ \times \Omega, R^n]$, $\tau_k \in C^1[\Omega, (0, \infty)]$, $\tau_k(x) < \tau_{k+1}(x)$ *for every* k, $\lim_{k \to \infty} \tau_k(x) = \infty$ *uniformly in* $x \in \Omega$, *and* $I_k \in C[\Omega, R^n]$;

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(ii)(a) $(\partial\tau_k(x)/\partial x) f(t, x) \leqslant 0$ *for* $(t, x) \in R_+ \times \Omega$, *and*

(b) $x + I_k(x) \in \Omega$ *for* $x \in \Omega$ *and* $((\partial\tau_k/\partial x)(x + sI_k(x)))\, I_k(x) \leqslant 0$, $0 \leqslant s \leqslant 1$, *for every* k.

Then, every solution of (1) *meets any given surface* S_j: $t = \tau_j(x)$ *at most once.*

Proof. Suppose the contrary. Then there exists a solution $x(t)$ of (1) and a surface S_j such that $x(t)$ meets S_j two or more times. Let the first hit be at $t = t_k$ for some k and another hit at $t = t^*$ so that we have

$$t_k = \tau_j(x(t_k)) \qquad \text{and} \qquad t^* = \tau_j(x(t^*)), \qquad t_0 < t_k < t^*.$$

The following two possible situations need consideration:

(A) $t^* = t_{k+m}$ and the solution $x(t)$ meets the surfaces S_j, $m-1$ times for $t \in (t_k, t_{k+m})$, $t_{k+1}, ..., t_{k+m-1}$, say;

(B) the solution $x(t)$ meets the surfaces S_j infinite number of times for $t \in (t_k, t^*)$.

Consider the case (A). Condition (iia) implies that $\tau_i(x(t))$ is nonincreasing in $(t_\eta, t_{\eta+1}]$ for any $i \geqslant 1$ and $k \leqslant \eta \leqslant k+m-1$, while (iib) shows that

$$\tau_i(x + I_i(x)) \leqslant \tau_i(x) \tag{2}$$

for any $x \in \Omega$ and $i \geqslant 1$. Suppose that $x(t)$ hits S_{n_i} at $t = t_i$, $i = k+1, k+2, ..., k+m-1$. We then have, letting $x_k = x(t_k)$,

(a) $t_k = \tau_j(x_k) \geqslant \tau_j(x_k + I_k(x_k)) \geqslant \tau_j(x_{k+1})$,

and for $k+1 \leqslant i \leqslant k+m-1$,

(b) $t_i = \tau_{n_i}(x_i) \geqslant \tau_{n_i}(x_i + I_{n_i}(x_i)) \geqslant \tau_{n_i}(x_{i+1})$.

It then follows from (a) that $j < n_{k+1}$, for otherwise, we are lead to the contradiction $t_{k+1} = \tau_{n_{k+1}}(x_{k+1}) \geqslant \tau_j(x_{k+1}) \geqslant t_k$. By repeating the same procedure and using (b), it is easy to conclude that $j < n_{k+1} < \cdots < n_{k+m-1} < j$, which is a contradiction.

In case (B), we can find a sequence of consecutive impulse moments t_k, $t_{k+1}, ...$ such that

$$t_k < t_{k+i} < t_{k+j} < t^* \qquad \text{for} \quad 1 \leqslant i \leqslant j. \tag{3}$$

We have again two possibilities. (B_1) $x(t)$ meets an infinite number of surfaces S_i different from each other at $\{t_{k+j}: j \geqslant 1\}$. ($\mathrm{B}_2$) For some $1 < j_1 < j_2$ and j, $x(t)$ meets the same surface S_j at j_1 and j_2.

In case (B_1), since $\lim_{k\to\infty}\tau_k(x)=\infty$ uniformly on Ω, there must be a j such that $\tau_j(x)>t^*$ on Ω, and i such that

$$t_{k+i}=\tau_j(x(t_{k+i}))>t^*,$$

a contradiction to (3). If (B_2) holds, we are back in situation (A) and hence it is also impossible. The proof is therefore complete.

Remark 1. The conclusion of Theorem 1 remains true when condition (ii) is replaced by

(ii*)(a) $(\partial\tau_k(x)/\partial x)\,f(t,x)\leqslant\alpha$, $0\leqslant\alpha\leqslant 1$, for $(t,x)\in R_+\times\Omega$;

(b) $x+I_k(x)\in\Omega$ for $x\in\Omega$ and $((\partial\tau_k/\partial x)(x+sI_k(x)))\,I_k(x)<0$, $0\leqslant s\leqslant 1$, for every k.

EXAMPLE. Consider the impulsive differential equation

$$x'=\cos t,\qquad t\neq\tau_k(x),\ x(0)=0,$$
$$\Delta x=I_k(x),\qquad t=\tau_k(x),$$

where $\tau_k(x)=-(x+1)+(2\pi+1)k$ and $I_k(x)=1$, for all k. Since

$$\frac{\partial\tau_k(x)}{\partial x}f(t,x)=-\cos t\leqslant 1\qquad\text{and}\qquad\left(\frac{\partial\tau_k(x+sI_k(x))}{\partial x}\right)I_k(x)=-1<0,$$

the assumptions of Theorem 1 with (ii*) are verified and hence there is no pulse phenomena. In fact, one can verify that the solutions meet the surfaces S_k consecutively at the points $(t,x)=(2k\pi,k-1)$, $k=1,2,\ldots$.

We shall next discuss a result which gives sufficient conditions for any solution to meet each surface exactly once.

THEOREM 2. *Assume that*

(i) $f\in C[R_+\times\Omega,R^n]$, $I_k\in C[\Omega,R^n]$, $\tau_k\in C^1[\Omega,(0,\infty)]$, $\tau_k(x)$ *is bounded, and* $\tau_k(x)<\tau_{k+1}(x)$, *for each* k;

(ii)(a) $(\partial\tau_k(x)/\partial x)\,f(t,x)\leqslant 1$, *for* $(t,x)\in R_+\times\Omega$;

(b) $((\partial\tau_k/\partial x)(x+sI_k(x)))\,I_k(x)<0$, *and*

(c) $((\partial\tau_k/\partial x)(x+sI_{k-1}(x)))\,I_{k-1}(x)\geqslant 0$, $0\leqslant s\leqslant 1$, $x+I_k(x)\in\Omega$ *whenever* $x\in\Omega$.

Then every solution $x(t)=x(t,t_0,x_0)$ *of* (1) *such that* $0\leqslant t_0<\tau_1(x_0)$ *meets each surface* S_k *exactly once.*

Proof. Let $x(t)=x(t,t_0,x_0)$ be any solution of (1) such that

$0 \leqslant t_0 < \tau_1(x_0)$. Since $\tau_1(x)$ is bounded and continuous on Ω, there is a unique $t_1 > t_0$ such that

$$t_1 = \tau_1(x(t_1)) \qquad \text{and} \qquad t < \tau_1(x(t)) \qquad \text{for} \quad t < t_1.$$

Hence $x(t)$ hits the surface S_1 at $t = t_1$.

Now setting $x_1 = x(t_1)$, $x_1^+ = x_1 + I_1(x_1)$, we obtain by (iib)

$$\tau_1(x_1) > \tau_1(x_1 + I_1(x_1)) = \tau_1(x_1^+).$$

On the other hand, (iic) implies that

$$\tau_2(x_1^+) = \tau_2(x_1 + I_1(x_1)) \geqslant \tau_2(x_1) > \tau_1(x_1),$$

which yields

$$\tau_1(x_1^+) < t_1 < \tau_2(x_1^+).$$

Proceeding as before, we can find a unique $t_2 > t_1$ such that

$$t_2 = \tau_2(x(t_2, t_1, x_1^+)) \qquad \text{and} \qquad t < \tau_2(x(t, t_1, x_1^+)) \qquad \text{for} \quad t_1 < t < t_2.$$

Condition (iia) implies that the function $T(t) = t - \tau_1(x(t, t_1, x_1^+))$ is nondecreasing in (t_1, t_2), and since $T(t_1) > 0$, we get

$$\tau_1(x(t, t_1, x_1^+)) < t \qquad \text{for} \quad t \in [t_1, t_2].$$

Therefore, $x(t)$ meets S_2 first at $t = t_2$ after t_1. Setting again $x_2 = x(t_2, t_1, x_1^+)$, $x_2^+ = x_2 + I_2(x_2)$, and using condition (ii), we can conclude that

$$\tau_2(x_2^+) < t_2 < \tau_3(x_2^+).$$

A similar argument as before yields a $t_3 > t_2$ such that $x(t)$ meets S_3 first at $t = t_3$ after t_2. Repeating this process, one can prove the stated claim and therefore the proof is complete.

We shall now obtain conditions for pulse phenomena to occur. First, we consider a simple situation where we have only one surface.

THEOREM 3. *Assume that*

(i) $f \in C[R_+ \times \Omega, R^n]$, $I \in C[\Omega, R^n]$, $\tau \in C^1(\Omega, (0, \infty)]$, *and* $\tau(x)$ *is bounded;*

(ii) $x + I(x) \in \Omega$ *for* $x \in \Omega$ *and* $(\partial\tau/\partial x)(x + sI(x))I(x) > 0$, $0 \leqslant s \leqslant 1$.

Then, every solution $x(t) = x(t, t_0, x_0)$ *such that* $0 \leqslant t_0 < \tau(x_0)$ *meets the surface* S: $t = \tau(x)$ *several times.*

Proof. In the present case, we have to consider the impulsive differential system

$$\begin{aligned} x' &= f(t, x), \ t \neq \tau(x), \qquad & x(t_0) = x_0, \ t_0 \geqslant 0 \\ \Delta x &= I(x), & t = \tau(x). \end{aligned} \tag{1*}$$

Let $x(t) = x(t, t_0, x_0)$ be any solution of (1*) such that $t_0 < \tau(x_0)$. Then, since $\tau(x)$ is bounded and continuous on Ω, we arrive at a $t_1 > t_0$ such that $t_1 = \tau(x(t_1))$ which shows that $x(t)$ hits the surface S at $t = t_1$. Now, setting $x_1 = x(t_1)$, $x_1^+ = x_1 + I(x_1)$, we have, because of (ii),

$$t_1 = \tau(x_1) < \tau(x_1 + I(x_1)) = \tau(x_1^+).$$

Let $x(t) = x(t, t_1, x_1^+)$ be any solution of (1*) starting at (t_1, x_1^+), and proceeding as before, we arrive at a $t_2 > t_1$ such that $t_2 = \tau(x(t_2))$. This implies that $x(t)$ meets the surface S a second time at $t = t_2$. This process can be continued as long as the solutions $x(t)$ remain in Ω and therefore the proof is complete.

The next result offers conditions for any solution to hit a given surface S_j several times.

THEOREM 4. *Assume that*

(i) $f \in C[R_+ \times \Omega, R^n]$, $I_k \in C[\Omega, R^n]$, $\tau_k \in C^1[\Omega, (0, \infty)]$, *and* $\tau_k(x) < \tau_{k+1}(x)$ for every k;

(ii) *for a fixed* j, $\tau_j(x)$ *is bounded and*

(a) $(\partial \tau_{j-1}(x)/\partial x) f(t, x) \leqslant 1$, *for* $(t, x) \in R_+ \times \Omega$;

(b) $x + I_j(x) \in \Omega$ *for* $x \in \Omega$, $(\partial \tau_{j-1}/\partial x)(x + sI_j(x)) I_j(x) > 0$, *and*

(c) $[(\partial \tau_{j-1}/\partial x)(x + sI_j(x))] I_j(x) \leqslant 0$ *for* $0 \leqslant s \leqslant 1$.

Then, every solution $x(t) = x(t, t_0, x_0)$ *of* (1) *such that* $\tau_{j-1}(x_0) < t_0 < \tau_j(x_0)$ *meets the surface* S_j *several times.*

Proof. Let $x(t) = x(t, t_0, x_0)$ be any solution of (1) such that $\tau_{j-1}(x_0) < t_0 < \tau_j(x_0)$. Since $\tau_j(x)$ is bounded and continuous on Ω, there is a unique $t_1 > t_0$ such that

$$t_1 = \tau_j(x(t_1)) \qquad \text{and} \qquad t < \tau_j(x(t)) \qquad \text{for all} \quad t_0 < t < t_1.$$

Condition (iia) shows that the function $T(t) = t - \tau_{j-1}(x(t))$ is nondecreasing in (t_0, t_1), and since $T(t_0) > 0$, we get

$$\tau_{j-1}(x(t)) < t \qquad \text{for} \quad t \in [t_0, t_1].$$

Hence $x(t)$ meets the surface S_j at $t = t_1$ before hitting any other surface.

Let $x_1 = x(t_1)$ and $x_1^+ = x_1 + I_j(x_1)$. Then by (iib), (iic) we have

$$t_1 = \tau_j(x_1) < \tau_j(x_1 + I_j(x_1)) = \tau_j(x_1^+),$$

and

$$\tau_{j-1}(x_1^+) = \tau_{j-1}(x_1 + I_j(x_1)) \leqslant \tau_{j-1}(x_1) < t_1.$$

Thus we arrive at

$$\tau_{j-1}(x_1^+) < t_1 < \tau_j(x_1^+).$$

Let $x(t) = x(t, t_1, x_1^+)$ be any solution of (1) starting at (t_1, x_1^+). Then proceeding as before, we obtain a $t_2 > t_1$ such that $t_2 = \tau_j(x(t_2))$. This shows that every solution $x(t)$ hits the surface S_j at least twice. We can now repeat the same argument and therefore the proof is complete.

The conclusion of Theorem 4 may also be stated as follows: every solution $x(t, t_0, x_0)$ which hits the surface S_j once, hits it several times.

The pulse phenomena can occur in many complicated ways. We shall now give a typical result in that direction.

THEOREM 5. *Assume that*

(i) $f \in C[R_+ \times \Omega, R^n]$, $I_k \in C[\Omega, R^n]$, $\tau_k \in C^1[\Omega, (0, \infty)]$, $\tau_k(x) < \tau_{k+1}(x)$, *and* $\tau_k(x)$ *is bounded for every* k;

(ii) $(\partial\tau_k(x)/\partial x) f(t, x) \leqslant 1$, *for* $(t, x) \in R_+ \times \Omega$, *for every* k;

(iii) $x + I_k \in \Omega$ *for* $x \in \Omega$ *and every* k, *and for some fixed* $k = k_0$,

(b$_1$) $((\partial\tau_{k_0}/\partial x)(x + sI_j(x)))\, I_j(x) \geqslant 0$ *for all* $j < k_0$ *and* $\tau_j(x) > \tau_{k_0-1}(x + I_j(x))$;

(b$_2$) $((\partial\tau_{k_0-1}/\partial x)(x + sI_j(x)))\, I_j(x) \leqslant 0$ *for all* $j > k_0$ *and* $\tau_j(x) < \tau_{k_0}(x + I_j(x))$, *where* $0 \leqslant s \leqslant 1$.

Then, every solution $x(t) = x(t, t_0, x_0)$ *of* (1) *meets the surface* S_{k_0} *several times.*

Proof. Let $x(t) = x(t, t_0, x_0)$ be any solution of (1). Since $\tau_k(x)$ is bounded and continuous on Ω, we arrive at a $t_1 > t_0$ such that $t_1 = \tau_j(x(t_1))$ for some j, that is, $x(t)$ meets the surface S_j at $t = t_1$. There are two cases to consider, namely $j < k_0$ or $j > k_0$. Suppose first that $j < k_0$. Then, letting $x_1 = x(t_1)$ so that $x_1^+ = x_1 + I_j(x_1)$, we get from (iiib$_1$) and the fact that $\tau_k(x) < \tau_{k+1}(x)$ for every k

$$\tau_j(x_1) < \tau_{k_0}(x_1) \leqslant \tau_{k_0}(x_1 + I_j(x_1)),$$

which implies

$$t_1 = \tau_j(x_1) < \tau_{k_0}(x_1 + I_j(x_1)) = \tau_{k_0}(x_1^+).$$

By (iiib$_1$), we also get

$$t_1 = \tau_j(x_1) < \tau_{k_0-1}(x_1 + I_j(x_1)) = \tau_{k_0-1}(x_1^+)$$

and therefore

$$\tau_{k_0-1}(x_1^+) \leqslant t_1 < \tau_{k_0}(x_1^+).$$

Setting $T(t) = t - \tau_{k_0}(x(t))$, where $x(t) = x(t, t_1, x_1^+)$ is any solution through (t_1, x_1^+) and proceeding as in Theorem 4, we arrive at a $t_2 > t_1$ such that $t_2 = \tau_{k_0}(x(t_2))$, implying that $x(t)$ meets the surface S_{k_0} at $t = t_2$.

If $j > k_0$, we use (iiib$_2$) to obtain

$$\tau_{k_0-1}(x_1^+) < t_1 < \tau_{k_0}(x_1^+).$$

Hence, setting $T(t) = t - \tau_{k_0}(x(t))$, where $x(t) = x(t, t_1, x_1^+)$ is any solution of (1) and proceeding as before, it follows that there exists a $t_2 > t_1$ such that $x(t)$ meets the surface S_{k_0} at $t = t_2$.

If $x(t)$ hits the surface S_{k_0} several times after $t = t_2$, we are done. If not, $x(t)$ encounters some surface S_i, $i \neq k_0$ at $t_3 > t_2$, because $\tau_k(x)$ is bounded on Ω for every k. Arguing as before, we can show that there exists a $t_4 > t_3$ at which $x(t)$ meets S_{k_0} again. This process can be continued as long as solutions exist and therefore, the desired result follows proving the theorem.

Remark 2. One can easily verify that all the results remain valid if the boundedness requirement of $\tau_k(x)$ on Ω is replaced by the following: there is a $p_k \in C[[t_0, \infty), R_+]$ such that $(\partial\tau_k/\partial t)(x(t)) \leqslant p_k(t)$, for $t \neq \tau_j(x(t))$ where $x(t)$ is any solution of (1.3.1) and

$$\int_{t_0}^{\infty} p_k(t)\, dt < \infty. \tag{4}$$

As an illustration, we state and prove the following result.

THEOREM 6. *Assume that*

(i) $f \in C[R_+ \times \Omega, R^n]$, $I_k \in C[\Omega, R^n]$, $\tau_k \in C^1[\Omega, (0, \infty)]$, $\tau_k(x) < \tau_{k+1}(x)$ *for each* k;

(ii)(a) $(\partial\tau_k/\partial x)(x + sI_k(x))\, I_k(x) < 0$,

(b) $(\partial\tau_k/\partial x)(x + sI_{k-1}(x)) I_{k-1}(x) \geqslant 0$, $\quad 0 \leqslant s \leqslant 1$, $\quad x + I_k(x) \in \Omega$ *whenever* $x \in \Omega$;

(iii) *$p_k \in L^1[R_+, R_+]$ such that $\int_t^\infty p_k(s)\, ds = \infty$ for any k and $t \geqslant t_0$, and*

$$\frac{\partial \tau_k(x)}{\partial x} f(t, x) + p_k(t) \leqslant 1 \qquad \textit{for} \quad (t, x) \in R_+ \times \Omega. \tag{5}$$

Then every solution $x(t) = x(t, t_0, x_0)$ of (1) such that $0 \leqslant t_0 < \tau_1(x_0)$ meets each surface S_k exactly once.

Proof. We follow precisely the procedure of the proof of Theorem 2. Clearly (5) implies condition (iia) of Theorem 2. We note that the boundedness of $\tau_k(x)$ in Ω in Theorem 2 was used only to argue that $t_1 < \tau_k(x_1^+)$ implies the existence of a $t_2 > t_1$, such that $t_2 = \tau_k(x(t_2, t_1, x_1^+))$. In the present situation, (5) achieves the same purpose. Evidently, (5) shows that

$$T(t) = t - \tau_k(x(t, t_1, x_1^+)$$

is nondecreasing for $t \geqslant t_1$ since $dT(t)/dt \geqslant p_k(t)$. Hence $\int_{t_1}^\infty p_k(s)\, ds = \infty$ implies that there is a $t_2 > t_1$ satisfying

$$t_2 = \tau_k(x(t_2, t_1, x_1^+).$$

The rest of the argument is the same that employed in the proof of Theorem 2 and therefore the proof is complete.

References

1. D. D. Bainov and A. B. Dishliev, Sufficient conditions for absence of beating in systems of differential equations with impulses, *Appl. Anal.* **18** (1984), 67–73.
2. N. A. Perestyuk and A. M. Samoilenko, "Differential Equations with Impulsive Effect," Central Publications, Kiev, 1987.